

Controlling spatiotemporal dynamics with time-delay feedback

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We suggest a spatially local feedback mechanism for stabilizing periodic orbits in spatially extended systems. Our method, which is based on a comparison between present and past states of the system, does not require the external generation of an ideal reference state and can suppress both absolute and convective instabilities. As an example, we analyze the complex Ginzburg-Landau equation in one dimension, showing how the time-delay feedback enlarges the stability domain for traveling waves. [S1063-651X(96)51407-8]

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A common situation encountered in the operation of physical systems or devices is that a useful solution of the equations of motion turns out to be unstable in a parameter regime of interest. In many cases the desired behavior is a steady state or a regular periodic motion, and the instability eventually leads to chaotic fluctuations which limit the system's performance. Thus one is led to explore the possible modifications of the system that render the desired motion stable.

Recently, there has been intense interest in the application of proportional feedback for stabilizing periodic orbits [1,2]. Since the orbit in question is a solution to the equations of motion, the stabilizing feedback signal vanishes when control is successful, so that all the desirable features of the uncontrolled system are retained [3]. Many methods for controlling systems with only a few relevant degrees of freedom have now been successfully demonstrated.

Some of the most interesting and significant dynamical instabilities arise in spatially extended systems which may be described by partial differential equations, a large number of coupled ordinary differential equations, or coupled map lattices. Well-known examples of practical interest include convecting fluids, large Fresnel number lasers, and arrays of semiconductor lasers. For small systems, the number of unstable modes remains small and techniques involving only a few degrees of freedom can effectively treat the spatiotemporal dynamics [4]. For the case of open systems with convective instabilities, control of larger systems has also been demonstrated [5].

In this paper, we present and analyze a method for stabilizing periodic orbits in arbitrarily large systems. (A different approach has been suggested by Hu and Qu [6].) Our approach is a generalization of the technique known as "extended time-delay autosynchronization" (ETDAS), which has been successfully applied to a variety of low-dimensional systems, both numerical and experimental [2]. In ETDAS, the current state of the system is compared to its state one or more periods in the past. In this paper, we take the feedback mechanism to be local in space, in the sense that the signal at a given point depends only on previous behavior of that particular point, not on the behavior of distant regions or spatially averaged quantities. The locally determined feedback is applied simultaneously at all points in the system. Our analytical treatment of the important special case of the one-dimensional (1D) complex Ginzburg-Landau

equation shows both that this method can stabilize spatially extended periodic orbits and, more generally, that the introduction of spatially local time-delayed interactions can dramatically alter the stability properties of extended deterministic systems.

Our method has three key features: First, it applies equally well to systems with absolute or convective instabilities. Second, because the feedback is locally determined, the method is scalable up to arbitrarily large system sizes with no increase in complexity. Third, it does not require comparison to an external reference signal and therefore might be implemented in fast (optical) systems, systems in which the reference state is not known *a priori*, or systems in which the reference state has nontrivial spatiotemporal structure. In systems where the control works, the only information that must be supplied by the controller is the period τ of the desired motion.

The general approach we take is as follows. To generate the feedback signal for a system described by an evolving field $\phi(x,t)$, the entire field is compared to time-delayed images of itself $\phi(x,t-n\tau)$, with τ chosen to be the period of the desired orbit and n taking all positive integer values. With $t_n \equiv t - n\tau$, the feedback signal is the field

$$\epsilon_\phi(x,t) = \gamma \sum_{n=0}^{\infty} R^n [\phi(x,t_n) - \phi(x,t_{n+1})], \quad (1)$$

where γ is a real parameter (the gain) and R is a real parameter between 0 and 1. We assume here that τ is known in advance. (The problem of finding appropriate periodic orbits and their periods is beyond the scope of this paper.) It is clear that ϵ_ϕ vanishes identically when $\phi(x,t)$ is periodic in time with period τ . As emphasized elsewhere [2], the infinite sum can be obtained in practice with a recursive feedback loop that contains only a single time-delay device. In some systems, it may be possible to implement this form of control directly, e.g., by using optical elements that preserve the spatial structure of a laser beam. Alternatively, $\epsilon_\phi(x,t)$ can be considered as a limiting case of the placement of independent ETDAS controllers at many discrete points in the system.

We are interested in the extent to which proper choices of γ and R can improve the stability of selected time-periodic patterns of the field $\phi(x,t)$ for arbitrarily large system sizes.

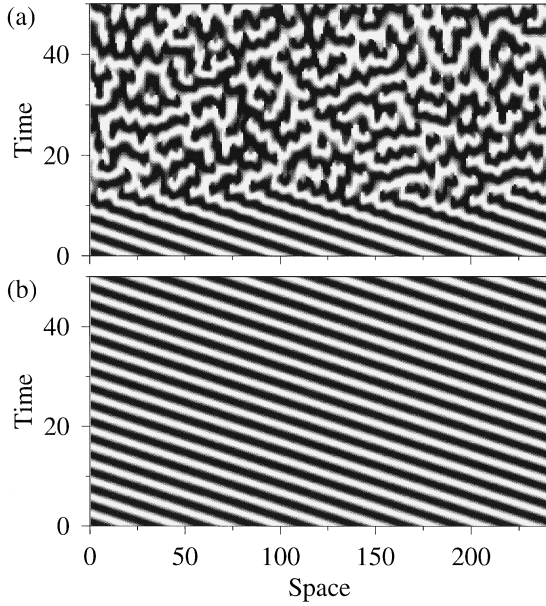


FIG. 1. Numerical simulation of the 1D CGL equation with periodic boundaries and parameters ($c_1=c_3=2$). (See final page of text for simulation details.) The plots show the evolution of the phase of the complex field A . Black (white) regions correspond to phases near 0 ($\pm\pi$). (a) Without control, an unstable traveling wave state (wave number $k=0.075\pi$) evolves into a turbulent state. (b) Time-delayed control stabilizes the traveling wave. The control parameters [see Eq. (1)] are $\gamma=-0.6$ and $R=0.75$.

In this paper, we treat the 1D complex Ginzburg-Landau (CGL) equation with a cubic nonlinearity, a partial differential equation that describes a large class of systems that undergo a bifurcation from regular oscillations to spatiotemporal chaos [7]. In addition to its relevance to fluid and laser systems, this equation has the advantage of possessing purely sinusoidal traveling wave solutions which permit a detailed analysis. Our linear stability analysis shows that periodic traveling wave states can indeed be stabilized by appropriate choices of γ and R , even for system parameters corresponding to turbulence (“defect chaos” [7]) in the uncontrolled equation. A numerical illustration of successful control is shown in Fig. 1.

The controlled CGL equation we study may be written in dimensionless form as

$$\partial_t A = A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)|A|^2 A + \epsilon_A, \quad (2)$$

where x is a one-dimensional continuous variable, $A(x,t)$ is a complex field, c_1 and c_3 are real parameters, and $\epsilon_A(x,t)$ is the control term defined above. Without the control term, Eq. (2) admits traveling wave solutions of wave number k and frequency $\omega = (c_1 + c_3)k^2 - c_3$. Each solution,

$$A_k(x,t) = \sqrt{1 - k^2} \exp(ikx - i\omega t), \quad (3)$$

becomes unstable for large enough c_1 and/or c_3 , and all of them are unstable for $c_1 c_3 > 1$ [9].

We find that when τ is chosen to be $2\pi/\omega$, the domain of c_1 and c_3 values over which the solution A_k is stable is expanded significantly for modest values of γ and R . Some typical results are shown in Fig. 2. Each panel of the figure

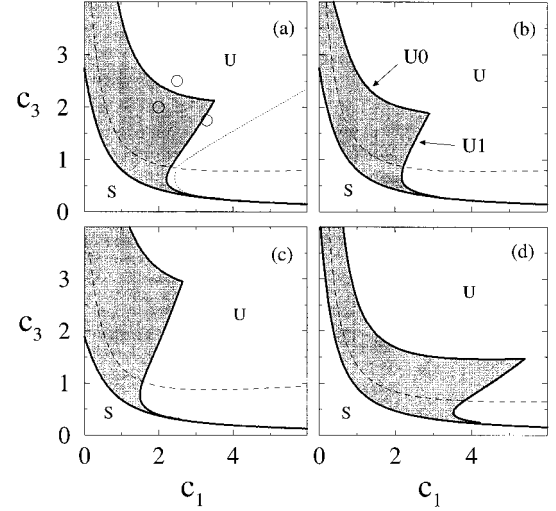


FIG. 2. Stability diagrams for three choices of wave number k and two choices of parameter R in Eq. (1). (a) $k=0.075\pi, R=0.75$, The circles mark the values of c_1, c_3 used in Fig. 3. (See final page of text for explanations of $U0, U1$, and the dotted line.) (b) $k=0.075\pi, R=0.5$, (c) $k=0.1\pi, R=0.75$, (d) $k=0.05\pi, R=0.75$. In the area labeled S the uncontrolled traveling wave is stable. In the shaded region, control is possible. In all cases shown here, control is achieved with gain $|\gamma| < 1$. In the region marked U , no value of γ can stabilize the traveling wave. The dashed line marks the transition from convective (lower left region) to absolute instability in the uncontrolled system.

corresponds to a different set of values for k and R . The shaded region represents the parameter values for which the traveling wave A_k is a stable solution of the equation with control, but would be unstable without control. The instability would be convective just above onset, but absolute for larger c_1 or c_3 [8], as indicated by the dashed line in Fig. 2. A surprising result is that time-delay feedback allows stabilization of traveling waves deep into the ordinarily chaotic, absolutely unstable regime ($c_1 c_3 \sim 8$), even though it is almost totally ineffective in stabilizing the uniform ($k=0$) oscillatory state.

We now describe our procedure for obtaining the stability domains depicted in Fig. 2. Standard linearization of Eq. (2) about $A_k(x,t)$ yields sets of ordinary, time-delay differential equations for the Fourier amplitudes of a perturbation. The technique of Ref. [10] is then applied to determine the stability of the different modes. In each period τ , a given mode grows or decays by a complex factor μ (a Floquet multiplier). A system is stable if and only if $|\mu| < 1$ for every mode. The defining relation for the Floquet multipliers of a general, finite-dimensional system controlled by ETDA is [10]

$$\left| \mu^{-1} T \left[\exp \int_0^\tau dt \left(\mathbf{J} + \gamma \frac{1 - \mu^{-1}}{1 - R\mu^{-1}} \mathbf{M} \right) \right] - \mathbf{1} \right| = 0, \quad (4)$$

where $T[\]$ represents the time-ordered product, \mathbf{J} is the Jacobian of the uncontrolled mode equations, \mathbf{M} is a “control matrix” that contains the information about the way in which the control signal is formed and enters into the dynamical equations, and $\mathbf{1}$ is the identity matrix.

In the present case, three features simplify the analysis: First, $\mathbf{M}=\mathbf{1}$; second, the Fourier modes decouple, with each yielding a condition of the form of Eq. (4) with 2×2 matrices \mathbf{J} and \mathbf{M} ; third, neither \mathbf{J} nor \mathbf{M} is time dependent. The latter is due first to the trivial time dependence of the desired solution A_k and second to the directly additive way in which ϵ appears in the equation for the controlled system. In such cases, the determinant in Eq. (4) may be evaluated explicitly by solving the differential equation that yields the time-ordered product. Here the defining relation for the Floquet multipliers associated with a perturbation at wave number $q+k$ becomes

$$g(\mu^{-1}) \equiv \mu^{-2} e^{2\alpha\tau} - 2\mu^{-1} e^{\alpha\tau} \cosh(\beta\tau) + 1 = 0, \quad (5)$$

$$\alpha = -q^2 - 2ic_1 k q + k^2 - 1 + \gamma \frac{1 - \mu^{-1}}{1 - R\mu^{-1}},$$

$$\beta = [(1 - k^2 - 4ic_3 k q + 2c_1 c_3 q^2)(1 - k^2) + 4k^2 q^2 + 4ic_1 k q^3 - c_1^2 q^4]^{1/2}.$$

The state $A_k(x,t)$ with particular choices of c_1 , c_3 , γ , and R is linearly stable if and only if all of the roots of g lie outside the unit circle. Note that $g(\mu^{-1})$ has an infinite number of roots due to the time delay in the system. As described in Ref. [10], it is straightforward to perform a winding number calculation (or a contour integration) that will return \mathcal{N} , the number of roots that lie inside the unit circle. The linear stability condition then reduces to $\mathcal{N}=0$. The winding number calculation is performed numerically by evaluating g on selected points on the unit circle. The precision of such a technique is determined by how well one can distinguish between a root inside the unit circle and one very close, but still outside. Using an adaptive step-size method, we resolved the location of such roots to an accuracy of 10^{-6} .

It is at this point that the many degrees of freedom in a spatially extended system complicate the analysis. In order for a particular state to be stable, it must be true that a single value of γ exists for which A_k is stable with respect to perturbations at all wavenumbers. To see whether such a γ exists for fixed k , c_1 , c_3 , and R , it is useful to plot the region of stability in the space of γ and the perturbation wave number q [11]. Figure 3(a) shows an example for which A_k is linearly stable against perturbations of all wave numbers for a range of γ (shown between the dashed lines). Note that the plot must be symmetric about $q=0$ since from Eq. (5) it is clear that $\mu(q) = \mu^*(-q)$. The rapid divergence of the stability boundaries for large q merely reflects the fact that the system is highly stable with respect to large q perturbations in the absence of control.

Figures 3(b) and 3(c) show why control cannot be achieved for some values of c_1 , c_3 , k , and R . The problem is that peaks in the lower boundary reach values of γ that are already ruled out by valleys in the upper boundary, so that no single value of γ can stabilize all wave numbers. The source of the peaks may be understood as follows: For a periodic state with frequency ω and \mathbf{J} and \mathbf{M} independent of time, it can be shown that ETDAS cannot stabilize a perturbation for which $\arg\mu = m\omega$, where m is any integer. The peaks in the lower boundary occur at wave numbers where this condition

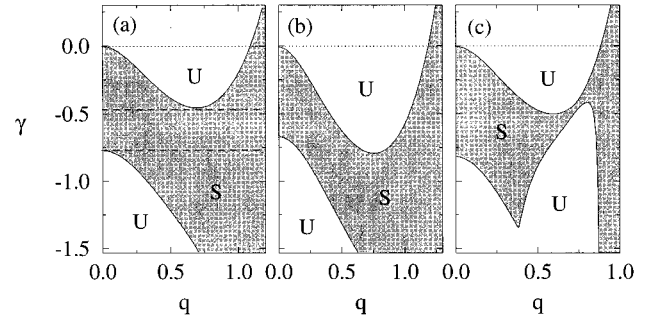


FIG. 3. Maps of the stable domains (shaded area) for $k=0.075\pi$ and $R=0.75$ in terms of the feedback gain γ and the perturbation wave number q for four values of (c_1, c_3) . (a) $c_1=c_3=2$, (b) $c_1=c_3=2.5$, and (c) $c_1=3.3, c_3=1.75$. The dotted lines are drawn at $\gamma=0$ to draw attention to the case of no control. In (a) the dashed lines show the range of γ for which control is successful.

is approximately satisfied. In Fig. 3, the peak in the lower boundary at $q=0$ corresponds to $m=0$. In Fig. 3(c), the peak at $q\sim 0.75$ corresponds to $m=\pm 1$. In the present case there can be no effect from higher $|m|$ because control will already have been lost due to $m=\pm 1$.

By analyzing stability diagrams in the q - γ plane for a grid of values in the c_1 - c_3 plane, one can construct the stability diagrams shown in Fig. 2. As in the example of Fig. 3(a), there is a range of the feedback gain γ that successfully achieves control for each point in the shaded area of Fig. 2. In general, the minimum value of $|\gamma|$ required increases smoothly as c_1 and/or c_3 increases. In the domains shown here, as well as others we investigated, the value of $|\gamma|$ required for stability is less than 1 even at the highest values of c_1 and c_3 in the controllable domain. (Details will be given in a longer paper.) Beyond the line labeled $U0$ ($U1$), control is lost due to the mechanism described above with $m=0$ ($m=\pm 1$), not through a divergence in the required γ .

Figures 2(a), 2(c), and 2(d) illustrate how the stability boundaries shift for different choices of k . Figure 2(a) corresponds to $k=0.075\pi$. For larger k [Fig. 2(c)], the $U0$ line moves farther from the uncontrolled stability boundary and $U1$ moves closer. For smaller k [Fig. 2(d)], the situation is reversed. As k is decreased toward 0, the boundary $U0$ approaches the original uncontrolled stability boundary, so that no enhancement of the $k=0$ state is obtainable. Figures 2(a) and 2(b) show the effect of changing R . As R is increased, the domain of stability increases in area. However, even as R approaches its maximum value of 1, the domain of stability cannot include the region in which one of the unstable modes of the uncontrolled system has frequency $\pm\omega$. The boundary of this region is the dotted line in Fig. 2(a).

We have checked specific aspects of the results presented in Fig. 2 with numerical simulations of the controlled CGL equation. Periodic boundary conditions were employed with the system size chosen to be an integer multiple of the wavelength of the traveling wave. System sizes corresponded to a length of at least $15\times 2\pi/k$. The simulations were performed with a second-order predictor-corrector and finite difference technique with time steps of order 10^{-2} and spatial resolution ~ 400 points. The instabilities were observed to

occur at values of (c_1, c_3) consistent with the analytic results presented here for infinite systems.

We have demonstrated that time-delay feedback can be effective in stabilizing periodic states of spatially extended systems. Application of this technique to the stabilization of unstable ordered states in fluid, laser, and biological systems is strongly suggested. We expect the control technique to be applicable to many types of periodic states, though the stability analysis may become complicated. If the linearized equation for the perturbations about the periodic solution has space-dependent coefficients but its time dependence is still trivial, perturbations can be decomposed into appropriate eigenfunctions and the analysis discussed here will apply. If the periodic state has trivial spatial dependence but nontrivial time dependence, then the stability of the Fourier modes can be analyzed using the numerical method of Ref. [10]. Finally, when the periodic state has complicated spatiotemporal structure, it appears that numerical integration of the controlled equations would be the most efficient approach. Even in the absence of any stability analysis, however, control can

be attempted in a physical system given only the knowledge of τ and the ability to adjust the single parameter γ .

Our work points to several important questions for future study. What is the minimum density of discrete controllers needed in situations where spatially continuous processing in the feedback loop is not possible? What level of noise can be tolerated? How can one force the system from the spatiotemporally chaotic state into the desired controllable state?

Finally, we suggest that the application of time-delayed feedback may be a valuable tool for studying the intrinsic physics of spatiotemporally chaotic systems. By varying γ and τ slowly, it may be possible to locate previously unknown periodic states or to observe other novel effects.

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